

# Maximum-norm error analysis of compact difference schemes for the backward fractional Feynman-Kac equation

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## Abstract

The fractional Feynman-Kac equations describe the distribution of functionals of non-Brownian motion, or anomalous diffusion, including two types called the forward and backward fractional Feynman-Kac equations, where the fractional substantial derivative is involved. This paper focuses on the more widely used backward version. Based on the discretized schemes for fractional substantial derivatives proposed recently, we construct compact finite difference schemes for the backward fractional Feynman-Kac equation, which has  $q$ -th ( $q = 1, 2, 3, 4$ ) order accuracy in temporal direction and fourth order accuracy in spatial direction, respectively. In the case  $q = 1$ , the numerical stability and convergence of the difference scheme in the discrete  $L^\infty$  norm are proved strictly, where a new inner product is defined for the theoretical analysis. Finally, numerical examples are provided to verify the effectiveness and accuracy of the algorithms.

**Keywords:** Backward fractional Feynman-Kac equation, Fractional substantial derivative, Compact scheme, Maximum-norm error analysis, Stability, Convergence

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## 1. Introduction

Diffusive motions exist widely in the nature, among the fields from condensed matter physics [1, 2, 3], to hydrodynamics [4], meteorology [5], and finance [6, 7]. Thus the Brownian functionals play important role in science community. Assume  $x(t)$  is a path of a Brownian particle in the time interval  $(0, t)$ , and  $U(x)$  is some prescribed function. Then a Brownian functional can be defined as  $A = \int_0^t U[x(\tau)]d\tau$ .  $A$  is a random variable for  $x(t)$  is a random path. On account of the diversity of the function  $U(x)$  being chosen, the Brownian functional  $A$  models different phenomena. In 1949, by using Feynman's path integral method Kac derived the (imaginary time) Schrödinger equation for the distribution function of  $A$  [8]. However, in recent years, by realizing that numerous anomalous diffusion phenomena exist widely in many systems, the scientists pay more and more attention to the anomalous diffusion processes, or the non-Brownian motion, which can be modeled more exactly by fractional differential equations.

Let  $x(t)$  be a trajectory of non-Brownian particle. The functional of anomalous diffusion has the same form as the Brownian functional

$$A = \int_0^t U[x(\tau)]d\tau. \quad (1.1)$$

For the different prescribed function  $U(x)$ , various physics processes can be characterized. For instance, when taking  $U(x) = 1$  in a given domain and to be zero otherwise,  $A$  models the time spent by a particle in the domain. The corresponding functional can be used in kinetic studies of chemical reactions that take place exclusively in the domain [9, 10]. When the motion of the particles is non-Brownian in dispersive systems with inhomogeneous disorder, we take  $U(x) = x$  or  $x^2$  [10]. By employing a versatile framework for describing the motion of particles in disordered systems, i.e., the continuous time random walk (CTRW), Carmi, Turgeman, and Barkai derived the forward and backward fractional Feynman-Kac equations [10, 11, 12], where the fractional substantial derivative is involved. Both forward and backward fractional Feynman-Kac

equations describe the distributions of functionals of the widely observed sub-diffusive processes. While in most cases, scholars are only interested in the distribution of the functional  $A$  and regardless of the final position of the particle,  $x$ , it turns out to be more convenient to use the backward version. The backward fractional Feynman-Kac equation which is shown as following, will be discussed detailedly in our work. Denote by  $P(x, A, t)$  the probability density function (PDF) of  $A$  at time  $t$ , given that the process has started at  $x$ . The backward fractional Feynman-Kac equation is given as [10, 11, 12]

$$\frac{\partial}{\partial t}P(x, \rho, t) = K_\alpha {}^sD_t^{1-\alpha} \frac{\partial^2}{\partial x^2}P(x, \rho, t) - \rho U(x)P(x, \rho, t), \quad (1.2)$$

where  $P(x, \rho, t) := \int_0^\infty P(x, A, t)e^{-\rho A}dA$ ,  $\Re(\rho) > 0$ ,  $U(x) \geq 0$ ; the functional  $A$  is defined as (1.1) and  $\alpha \in (0, 1)$ ; the diffusion coefficient  $K_\alpha$  is a positive constant, and the symbol  ${}^sD_t^\nu$  represents the Friedrich's fractional substantial derivative of order  $\nu$  [13].

For the past few years, the numerical methods for solving fractional partial differential equations (PDEs) have been well developed, including finite difference methods [14, 15, 16, 17, 18], finite element methods [19, 20, 21], and spectral methods [22, 23], etc. However, for the PDEs with fractional substantial derivative, though there have been some works for getting the numerical solutions [24], high order finite difference schemes with the maximum norm error estimates are still scarce. As is well-known, high order schemes lead to more accurate results if the solution of the equation is regular enough. The fractional substantial derivative is a non-local time-space coupled operator, which makes numerically solving the corresponding equations more difficult than other fractional PDEs, especially when using high order schemes. Besides, compared with the error estimates in discrete  $L^2$  norm, the discrete  $L^\infty$  norm error estimates provide more immediate insight on the error occurring during time evolution. Thus, in practice, error estimates in the grid independent maximum norm are preferred in numerical analysis. The purpose of this paper is to develop high order compact difference schemes for the backward fractional Feynman-Kac equation and provide a rigorous error analysis in the discrete  $L^\infty$  norm by the strategy of

introducing some kind of new inner product and corresponding norms.

The definitions of fractional substantial calculus are given as follows [25].

**Definition 1.1.** Let  $\nu > 0$ ,  $\rho$  be a constant, and  $P(t)$  be piecewise continuous on  $(0, \infty)$  and integrable on any finite subinterval of  $[0, \infty)$ . Then the fractional substantial integral of  $P(t)$  of order  $\nu$  is defined as

$${}^s I_t^\nu P(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t - \tau)^{\nu-1} e^{-\rho U(x)(t-\tau)} P(\tau) d\tau, \quad t > 0, \quad (1.3)$$

where  $U(x)$  is a prescribed function in (1.1).

**Definition 1.2.** Let  $\mu > 0$ ,  $\rho$  be a constant, and  $P(t)$  be  $(m - 1)$ -times continuously differentiable on  $(0, \infty)$  and its  $m$ -times derivative be integrable on any finite subinterval of  $[0, \infty)$ , where  $m$  is the smallest integer that exceeds  $\mu$ . Then the fractional substantial derivative of  $P(t)$  of order  $\mu$  is defined as

$${}^s D_t^\mu P(t) = {}^s D_t^m [{}^s I_t^{m-\mu} P(t)], \quad (1.4)$$

where

$${}^s D_t^m = \left( \frac{\partial}{\partial t} + \rho U(x) \right)^m. \quad (1.5)$$

According to the definition of fractional substantial derivative, equation (1.2) can be expressed as the following form

$${}^s D_t^1 P(x, \rho, t) = {}^s D_t^{1-\alpha} \left[ K_\alpha \frac{\partial^2}{\partial x^2} P(x, \rho, t) \right]. \quad (1.6)$$

Denote by  ${}_c D_t^\alpha$  the Caputo fractional substantial derivative [25], i.e.,

$$\begin{aligned} {}_c D_t^\alpha P(x, t) &:= {}^s D_t^\alpha \left[ P(x, t) - e^{-\rho U(x)t} P(x, 0) \right] \\ &= {}^s D_t^\alpha P(x, t) - \frac{t^{-\alpha} e^{-\rho U(x)t}}{\Gamma(1-\alpha)} P(x, 0), \end{aligned}$$

then the equivalent form of (1.6) can be written as [24]

$${}_c D_t^\alpha P(x, t) = K_\alpha \frac{\partial^2}{\partial x^2} P(x, t), \quad (1.7)$$

where  $P(x, \rho, t)$  is replaced by  $P(x, t)$  since  $\rho$  is given as a fixed constant. In the following, we still use  $P(x, t)$  for convenience.

The remainder of this paper is organized as follows. In Section 2, we construct the compact finite difference schemes for the backward fractional Feynman-Kac equation. In Section 3, by introducing some kind of inner product and norms, we prove the stability and convergence of the first order time discretization scheme in the discrete  $L^\infty$  norm rigorously. Numerical examples are provided to verify the effectiveness and accuracy of the proposed compact schemes from first to fourth order time discretization in Section 4. Finally we draw some conclusions in the last section.

## 2. Compact finite difference schemes for the backward fractional Feynman-Kac equation

This section focuses on deriving compact finite difference schemes for the backward fractional Feynman-Kac equation, which are of  $q$ -th ( $q = 1, 2, 3, 4$ ) order approximation in temporal direction and fourth order approximation in spatial direction, respectively.

Without loss of generality, consider the following backward fractional Feynman-Kac equation with non-homogeneous source term in the interval  $\Omega = (0, l)$ ,

$$\begin{aligned} {}^s D_t^\alpha P(x, t) &:= {}^s D_t^\alpha \left[ P(x, t) - e^{-\rho U(x)t} P(x, 0) \right] = K_\alpha \frac{\partial^2}{\partial x^2} P(x, t) + f(x, t), \\ 0 &< t \leq T, \quad x \in \Omega, \end{aligned} \tag{2.1}$$

and the initial and boundary conditions are given as

$$P(x, 0) = \varphi(x), \quad x \in \Omega, \tag{2.2}$$

$$P(0, t) = \psi_1(t), \quad P(l, t) = \psi_2(t), \quad 0 < t \leq T. \tag{2.3}$$

Let  $M, N$  be two positive integers, and  $h = l/M, \tau = T/N$  be the uniform size of spatial grid and time step, respectively. Then a spatial and temporal partition can be defined as  $x_i = ih$  for  $i = 0, 1, \dots, M$ , and  $t_n = n\tau$  for  $n = 0, 1, \dots, N$ . Denote  $\Omega_h = \{x_i \mid 0 \leq i \leq M\}$  and  $\Omega_\tau = \{t_n \mid 0 \leq n \leq N\}$ . Take  $\mathcal{V}_h = \{u \mid u = (u_0, u_1, \dots, u_M), u_0 = u_M = 0\}$  as grid function space on  $\Omega_h$ .

Then for any grid function  $u \in \mathcal{V}_h$ , we list the following notations

$$\begin{aligned}\delta_x u_{i-\frac{1}{2}} &= \frac{1}{h}(u_i - u_{i-1}), \quad \delta_x^2 u_i = \frac{1}{h}(\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}), \\ \mathcal{H}_h u_i &= \begin{cases} \frac{1}{12}(u_{i+1} + 10u_i + u_{i-1}), & 1 \leq i \leq M-1, \\ u_i, & i = 0 \text{ or } M. \end{cases}\end{aligned}$$

It is obvious that  $\mathcal{H}_h u_i = (1 + \frac{h^2}{12}\delta_x^2)u_i$  for  $1 \leq i \leq M-1$ . For any  $u, v \in \mathcal{V}_h$ , we introduce the inner product and norms as follows

$$\begin{aligned}\langle u, v \rangle &= h \sum_{i=0}^{M-1} (\delta_x u_{i+\frac{1}{2}})(\delta_x \bar{v}_{i+\frac{1}{2}}) - \frac{h^2}{12} h \sum_{i=1}^{M-1} (\delta_x^2 u_i)(\delta_x^2 \bar{v}_i), \\ \|u\|_\infty &= \max_{1 \leq i \leq M-1} |u_i|, \quad \|\delta_x u\| = \sqrt{h \sum_{i=1}^M |\delta_x u_{i-\frac{1}{2}}|^2}, \\ \|\delta_x^2 u\| &= \sqrt{h \sum_{i=1}^{M-1} |\delta_x^2 u_i|^2}, \quad \|u\| = \sqrt{h \sum_{i=1}^{M-1} |u_i|^2}.\end{aligned}$$

**Lemma 2.1.** *For  $\forall u \in \mathcal{V}_h$ , we have*

$$\frac{2}{3}\|\delta_x u\|^2 \leq \langle u, u \rangle \leq \|\delta_x u\|^2. \quad (2.4)$$

*Proof.* From the definition of inner product  $\langle u, v \rangle$ , we have  $\langle u, u \rangle = \|\delta_x u\|^2 - \frac{h^2}{12}\|\delta_x^2 u\|^2$ , which shows  $\langle u, u \rangle \leq \|\delta_x u\|^2$  immediately.

Since

$$h^2\|\delta_x^2 u\|^2 = h \sum_{i=1}^{M-1} |\delta_x u_{i+\frac{1}{2}} - \delta_x u_{i-\frac{1}{2}}|^2 \leq 2h \sum_{i=1}^{M-1} (|\delta_x u_{i+\frac{1}{2}}|^2 + |\delta_x u_{i-\frac{1}{2}}|^2) \leq 4\|\delta_x u\|^2,$$

we conclude that (2.4) holds.  $\square$

**Lemma 2.2.** [26] *For  $\forall u \in \mathcal{V}_h$ ,  $\|u\|_\infty \leq \frac{\sqrt{I}}{2}\|\delta_x u\|$ .*

**Lemma 2.3.** [27] *Let function  $g(x) \in C^6[a, b]$  and  $\xi(\lambda) = 5(1-\lambda)^3 - 3(1-\lambda)^5$ .*

*Then*

$$\mathcal{H}_h g''(x_i) = \delta_x^2 g(x_i) + \frac{h^4}{360} \int_0^1 [g^{(6)}(x_i - \lambda h) + g^{(6)}(x_i + \lambda h)] \xi(\lambda) d\lambda, \quad 1 \leq i \leq M-1.$$

According to [25], fractional substantial derivatives appeared in (2.1) have  $q$ -th order approximations, i.e.,

$$\begin{aligned} {}^s D_t^\alpha P(x, t)|_{(x_i, t_n)} &= \frac{1}{\tau^\alpha} \sum_{k=0}^n d_{i,k}^{q,\alpha} P(x_i, t_{n-k}) + \mathcal{O}(\tau^q), \\ {}^s D_t^\alpha [e^{-\rho U(x)t} P(x, 0)]|_{(x_i, t_n)} &= \frac{1}{\tau^\alpha} \sum_{k=0}^n d_{i,k}^{q,\alpha} e^{-\rho U_i(n-k)\tau} P(x_i, 0) + \mathcal{O}(\tau^q), \end{aligned} \quad (2.5)$$

where

$$U_i = U(x_i)$$

and

$$d_{i,k}^{q,\alpha} = e^{-\rho U_i k \tau} l_k^{q,\alpha}, \quad q = 1, 2, 3, 4. \quad (2.6)$$

$l_k^{1,\alpha}, l_k^{2,\alpha}, l_k^{3,\alpha}$  and  $l_k^{4,\alpha}$  are defined by (2.2), (2.4), (2.6) and (2.8) in [28], respectively. We denote  $(R_t^\alpha)_i^n = \mathcal{O}(\tau^q)$ .

Consider equation (2.1) at the point  $(x_i, t_n)$ , and we write it as following

$${}_c D_t^\alpha P(x_i, t_n) = K_\alpha \frac{\partial^2 P(x_i, t_n)}{\partial x^2} + f(x_i, t_n).$$

Acting the compact operator  $\mathcal{H}_h$  on both sides of the equation above, we have

$$\mathcal{H}_h({}_c D_t^\alpha P(x_i, t_n)) = K_\alpha \mathcal{H}_h \frac{\partial^2 P(x_i, t_n)}{\partial x^2} + \mathcal{H}_h f(x_i, t_n). \quad (2.7)$$

Assuming  $u(x, t) \in C_{6,2}^{x,t}([0, l] \times [0, T])$ , by use of Lemma 2.3 and (2.5) we obtain

$$\begin{aligned} &\mathcal{H}_h \left( \frac{1}{\tau^\alpha} \sum_{k=0}^n d_{i,k}^{q,\alpha} P(x_i, t_{n-k}) - \frac{1}{\tau^\alpha} \sum_{k=0}^n d_{i,k}^{q,\alpha} e^{-\rho U_i(n-k)\tau} P(x_i, 0) \right) \\ &= K_\alpha \delta_x^2 P(x_i, t_n) + \mathcal{H}_h f(x_i, t_n) + R_i^n, \end{aligned} \quad (2.8)$$

where

$$R_i^n = \mathcal{H}_h(R_t^\alpha)_i^n + K_\alpha (R_x)_i^n$$

with

$$(R_x)_i^n = \frac{h^4}{360} \int_0^1 \left[ \frac{\partial^6 P}{\partial x^6}(x_i - \lambda h, t_n) + \frac{\partial^6 P}{\partial x^6}(x_i + \lambda h, t_n) \right] \xi(\lambda) d\lambda.$$

Then there exists a constant  $\tilde{C}$  such that

$$|R_i^n| \leq \tilde{C}(\tau^q + h^4), \quad q = 1, 2, 3, 4. \quad (2.9)$$

Denote by  $P_i^n$  the approximated value of  $P(x_i, t_n)$ , and  $f_i^n = f(x_i, t_n)$ . Multiplying (2.8) by  $\tau^\alpha$ , and omitting the small term, we derive the compact finite difference schemes for solving the backward Feynman-Kac equation (2.1) with the initial condition (2.2) and boundary condition (2.3) as follows:

$$\begin{aligned} & \mathcal{H}_h \sum_{k=0}^n d_{i,k}^{q,\alpha} P_i^{n-k} - \mathcal{H}_h \sum_{k=0}^n d_{i,k}^{q,\alpha} e^{-\rho U_i(n-k)\tau} P_i^0 \\ &= K_\alpha \tau^\alpha \delta_x^2 P_i^n + \tau^\alpha \mathcal{H}_h f_i^n, \quad 1 \leq i \leq M-1, \quad 1 \leq n \leq N, \end{aligned} \quad (2.10)$$

$$P_i^0 = \varphi(x_i), \quad 1 \leq i \leq M-1, \quad (2.11)$$

$$P_0^n = \psi_1(t_n), \quad P_M^n = \psi_2(t_n), \quad 1 \leq n \leq N, \quad (2.12)$$

$$q = 1, 2, 3, 4.$$

To execute the procedure, we rewrite equation (2.10) as the following equivalent form

$$\begin{aligned} & \mathcal{H}_h d_{i,0}^{q,\alpha} P_i^n - \frac{K_\alpha \tau^\alpha}{h^2} (P_{i+1}^n - 2P_i^n + P_{i-1}^n) \\ &= \sum_{k=0}^{n-1} \mathcal{H}_h d_{i,k}^{q,\alpha} e^{-\rho U_i(n-k)\tau} P_i^0 - \sum_{k=1}^{n-1} \mathcal{H}_h d_{i,k}^{q,\alpha} P_i^{n-k} + \tau^\alpha \mathcal{H}_h f_i^0, \end{aligned} \quad (2.13)$$

$$q = 1, 2, 3, 4$$

with  $i = 1, 2, \dots, M-1$ . It is necessary to point out that when  $n = 1$ , the second term on the right hand side of (2.13) vanishes automatically.

### 3. Stability and convergence analysis of the difference scheme

In this section, we restrict  $U(x) = 1$  and do the detailed theoretical analysis for the first order discretization in temporal direction of schemes (2.10)–(2.12).

In the following, we introduce some lemmas first, and then prove the scheme is unconditionally stable and convergent in discrete  $L^\infty$  norm. For the simplification, we denote  $d_{i,k}^{1,\alpha}$  as  $d_{i,k}$  and  $l_k^{1,\alpha}$  as  $l_k$ , respectively.

**Lemma 3.1.** [24] *The coefficients  $l_k$  defined by (2.2) in [28] satisfy*

$$l_0 = 1; \quad l_k < 0, (k \geq 1); \quad \sum_{k=0}^{n-1} l_k > 0; \quad \sum_{k=0}^{\infty} l_k = 0; \quad (3.1)$$



and

$$\frac{1}{n^\alpha \Gamma(1-\alpha)} < \sum_{k=0}^{n-1} l_k = - \sum_{k=n}^{\infty} l_k \leq \frac{1}{n^\alpha}, \quad n \geq 1. \quad (3.2)$$

**Theorem 3.1.** *The difference scheme (2.10)–(2.12) is unconditionally stable with the assumption  $\Re(\rho) > 0$ .*

*Proof.* Assume  $\tilde{P}_i^n$  is the approximate solution of  $P_i^n$ , which is the exact solution of the scheme (2.10)–(2.12). Let  $\varepsilon_i^n = \tilde{P}_i^n - P_i^n$ ,  $0 \leq i \leq M$ ,  $0 \leq n \leq N$ . From (2.10)–(2.12), we have the perturbation error equations

$$\begin{aligned} \mathcal{H}_h \left( d_{i,0} \varepsilon_i^n + \sum_{k=1}^{n-1} d_{i,k} \varepsilon_i^{n-k} - \sum_{k=0}^{n-1} d_{i,k} e^{-\rho(n-k)\tau} \varepsilon_i^0 \right) &= K_\alpha \tau^\alpha \delta_x^2 \varepsilon_i^n, \\ 1 \leq i \leq M-1, \quad 1 \leq n \leq N, \end{aligned} \quad (3.3)$$

$$\varepsilon_0^n = \varepsilon_M^n = 0, \quad 1 \leq n \leq N. \quad (3.4)$$

By use of (2.6), equation (3.3) can also be written as

$$\mathcal{H}_h \left( l_0 \varepsilon_i^n + \sum_{k=1}^{n-1} e^{-\rho k \tau} l_k \varepsilon_i^{n-k} - \sum_{k=0}^{n-1} e^{-\rho n \tau} l_k \varepsilon_i^0 \right) = K_\alpha \tau^\alpha \delta_x^2 \varepsilon_i^n. \quad (3.5)$$

Multiplying (3.5) by  $h(-\delta_x^2 \bar{\varepsilon}_i^n)$  and summing up for  $i$  from 1 to  $M-1$ , we get

$$\begin{aligned} & h \sum_{i=1}^{M-1} (-\delta_x^2 \bar{\varepsilon}_i^n) \left[ \left( 1 + \frac{h^2}{12} \delta_x^2 \right) \left( l_0 \varepsilon_i^n + \sum_{k=1}^{n-1} e^{-\rho k \tau} l_k \varepsilon_i^{n-k} - \sum_{k=0}^{n-1} e^{-\rho n \tau} l_k \varepsilon_i^0 \right) \right] \\ &= h \sum_{i=1}^{M-1} (-K_\alpha \tau^\alpha) |\delta_x^2 \varepsilon_i^n|^2. \end{aligned} \quad (3.6)$$

Using the summation formula by parts and noticing (3.4), we obtain

$$\begin{aligned}
& l_0 h \sum_{i=0}^{M-1} \left| \delta_x \varepsilon_{i+\frac{1}{2}}^n \right|^2 - l_0 \frac{h^2}{12} h \sum_{i=1}^{M-1} \left| \delta_x^2 \varepsilon_i^n \right|^2 \\
&= -h \sum_{i=0}^{M-1} \sum_{k=1}^{n-1} l_k \left( \delta_x \varepsilon_{i+\frac{1}{2}}^n \right) \left( \delta_x \left( e^{-\rho k \tau} \varepsilon_{i+\frac{1}{2}}^{n-k} \right) \right) \\
&+ \frac{h^2}{12} h \sum_{i=1}^{M-1} \sum_{k=1}^{n-1} l_k \left( \delta_x^2 \varepsilon_i^n \right) \left( \delta_x^2 \left( e^{-\rho k \tau} \varepsilon_i^{n-k} \right) \right) \\
&+ h \sum_{i=0}^{M-1} \sum_{k=0}^{n-1} l_k \left( \delta_x \varepsilon_{i+\frac{1}{2}}^n \right) \left( \delta_x \left( e^{-\rho n \tau} \varepsilon_{i+\frac{1}{2}}^0 \right) \right) \\
&- \frac{h^2}{12} h \sum_{i=1}^{M-1} \sum_{k=0}^{n-1} l_k \left( \delta_x^2 \varepsilon_i^n \right) \left( \delta_x^2 \left( e^{-\rho n \tau} \varepsilon_i^0 \right) \right) - K_\alpha \tau^\alpha \| \delta_x^2 \varepsilon^n \|^2.
\end{aligned}$$

Then it can be deduced immediately that the inequality below holds.

$$l_0 \langle \varepsilon^n, \varepsilon^n \rangle \leq - \sum_{k=1}^{n-1} l_k \left| \langle e^{-\rho k \tau} \varepsilon^{n-k}, \varepsilon^n \rangle \right| + \sum_{k=0}^{n-1} l_k \left| \langle e^{-\rho n \tau} \varepsilon^0, \varepsilon^n \rangle \right|.$$

Let

$$A_1 = - \sum_{k=1}^{n-1} l_k \left| \langle e^{-\rho k \tau} \varepsilon^{n-k}, \varepsilon^n \rangle \right|$$

and

$$A_2 = \sum_{k=0}^{n-1} l_k \left| \langle e^{-\rho n \tau} \varepsilon^0, \varepsilon^n \rangle \right|.$$

From the Cauchy-Schwarz inequality and Lemma 3.1, we have the estimates

$$A_1 \leq \frac{1}{2} \sum_{k=1}^{n-1} (-l_k) \left( \langle \varepsilon^n, \varepsilon^n \rangle + \langle e^{-\rho k \tau} \varepsilon^{n-k}, e^{-\rho k \tau} \varepsilon^{n-k} \rangle \right),$$

$$A_2 \leq \frac{1}{2} \sum_{k=0}^{n-1} l_k \left( \langle \varepsilon^n, \varepsilon^n \rangle + \langle e^{-\rho n \tau} \varepsilon^0, e^{-\rho n \tau} \varepsilon^0 \rangle \right).$$

It follows that

$$\frac{1}{2} l_0 \langle \varepsilon^n, \varepsilon^n \rangle \leq -\frac{1}{2} \sum_{k=1}^{n-1} l_k \langle \varepsilon^{n-k}, \varepsilon^{n-k} \rangle + \frac{1}{2} l_0 \langle \varepsilon^0, \varepsilon^0 \rangle + \frac{1}{2} \sum_{k=1}^{n-1} l_k \langle \varepsilon^0, \varepsilon^0 \rangle. \quad (3.7)$$

Next, we prove

$$\langle \varepsilon^n, \varepsilon^n \rangle \leq \langle \varepsilon^0, \varepsilon^0 \rangle \quad (3.8)$$

by mathematical induction. In the case  $n = 1$ , (3.8) holds obviously according to (3.7). Suppose that for  $s = 1, 2, \dots, n-1$ ,

$$\langle \varepsilon^s, \varepsilon^s \rangle \leq \langle \varepsilon^0, \varepsilon^0 \rangle \quad (3.9)$$

holds. When  $s = n$ , according to (3.7) and (3.9), we have

$$\frac{1}{2} l_0 \langle \varepsilon^n, \varepsilon^n \rangle \leq -\frac{1}{2} \sum_{k=1}^{n-1} l_k \langle \varepsilon^0, \varepsilon^0 \rangle + \frac{1}{2} l_0 \langle \varepsilon^0, \varepsilon^0 \rangle + \frac{1}{2} \sum_{k=1}^{n-1} l_k \langle \varepsilon^0, \varepsilon^0 \rangle,$$

which indicates that  $\langle \varepsilon^n, \varepsilon^n \rangle \leq \langle \varepsilon^0, \varepsilon^0 \rangle$ .

Combining (3.8) with Lemma 2.1 and Lemma 2.2, we conclude that

$$\| \varepsilon^n \|_\infty^2 \leq \frac{l}{4} \|\delta_x \varepsilon^n\|^2 \leq \frac{l}{4} \cdot \frac{3}{2} \langle \varepsilon^n, \varepsilon^n \rangle \leq \frac{3l}{8} \langle \varepsilon^0, \varepsilon^0 \rangle \leq \frac{3l}{8} \|\delta_x \varepsilon^0\|^2,$$

which completes the proof.  $\square$

**Theorem 3.2.** *Let  $P_i^n$  be the solution of the difference scheme (2.10)–(2.12), and  $P(x_i, t_n)$  be the solution of the problem (2.1)–(2.3) with the assumption  $\Re(\rho) > 0$ . Denote  $E_i^n = P(x_i, t_n) - P_i^n$ ,  $0 \leq i \leq M$ ,  $0 \leq n \leq N$ , then there exists a positive constant  $C$  such that*

$$\| E^n \|_\infty \leq C (\tau + h^4), \quad 0 \leq n \leq N. \quad (3.10)$$

*Proof.* According to (2.8) and (2.10)–(2.12), we get the error equations

$$\begin{aligned} \mathcal{H}_h \left( d_{i,0} E_i^n + \sum_{k=1}^{n-1} d_{i,k} E_i^{n-k} \right) &= K_\alpha \tau^\alpha \delta_x^2 E_i^n + \tau^\alpha R_i^n, \\ 1 \leq i \leq M-1, \quad 1 \leq k \leq N, \end{aligned} \quad (3.11)$$

$$E_0^n = 0, \quad E_M^n = 0, \quad 1 \leq n \leq N, \quad (3.12)$$

$$E_i^0 = 0, \quad 1 \leq i \leq M-1 \quad (3.13)$$

with  $|R_i^n| \leq \tilde{C}(\tau + h^4)$  given in (2.9). Multiplying (3.11) by  $h(-\delta_x^2 \overline{E}_i^n)$  and summing up for  $i$  from 1 to  $M-1$ , we have

$$\begin{aligned} & h \sum_{i=1}^{M-1} (-\delta_x^2 \overline{E}_i^n) \left[ \left( 1 + \frac{h^2}{12} \delta_x^2 \right) \left( d_{i,0} E_i^n + \sum_{k=1}^{n-1} d_{i,k} E_i^{n-k} \right) \right] \\ &= h K_\alpha \tau^\alpha \sum_{i=1}^{M-1} (-\delta_x^2 \overline{E}_i^n) (\delta_x^2 E_i^n) + h \tau^\alpha \sum_{i=1}^{M-1} (-\delta_x^2 \overline{E}_i^n) R_i^n. \end{aligned} \quad (3.14)$$

From (2.6), (3.14) can also be written as

$$\begin{aligned}
& h \sum_{i=1}^{M-1} \left( -\delta_x^2 \overline{E}_i^n \right) \left( l_0 E_i^n + \sum_{k=1}^{n-1} e^{-\rho k \tau} l_k E_i^{n-k} \right) \\
& + \frac{h^2}{12} \sum_{i=1}^{M-1} \left( -\delta_x^2 \overline{E}_i^n \right) \left( l_0 \delta_x^2 E_i^n + \sum_{k=1}^{n-1} l_k \delta_x^2 (e^{-\rho k \tau} E_i^{n-k}) \right) \\
& = -K_\alpha \tau^\alpha h \sum_{i=1}^{M-1} |\delta_x^2 E_i^n|^2 - \tau^\alpha h \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i^n \right) R_i^n.
\end{aligned} \tag{3.15}$$

Using the summation formula by parts, we obtain from (3.15)

$$\begin{aligned}
& l_0 h \sum_{i=0}^{M-1} \left| \delta_x E_{i+\frac{1}{2}}^n \right|^2 + h \sum_{k=1}^{n-1} l_k \sum_{i=0}^{M-1} \left( \delta_x \overline{E}_{i+\frac{1}{2}}^n \right) \left( \delta_x e^{-\rho k \tau} E_{i+\frac{1}{2}}^{n-k} \right) \\
& - l_0 \frac{h^2}{12} \sum_{i=1}^{M-1} |\delta_x^2 E_i^n|^2 - \frac{h^2}{12} \sum_{k=1}^{n-1} l_k \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i^n \right) \left( \delta_x^2 e^{-\rho k \tau} E_i^{n-k} \right) \\
& = -K_\alpha \tau^\alpha h \sum_{i=1}^{M-1} |\delta_x^2 E_i^n|^2 - \tau^\alpha h \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i^n \right) R_i^n,
\end{aligned}$$

which implies

$$l_0 \langle E^n, E^n \rangle + \sum_{k=1}^{n-1} l_k \langle e^{-\rho k \tau} E^{n-k}, E^n \rangle = -K_\alpha \tau^\alpha \| \delta_x^2 E^n \|^2 - \tau^\alpha h \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i^n \right) R_i^n.$$

Then it can be deduced that

$$\begin{aligned}
l_0 \langle E^n, E^n \rangle &= \left| - \sum_{k=1}^{n-1} l_k \langle e^{-\rho k \tau} E^{n-k}, E^n \rangle - \tau^\alpha h \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i^n \right) R_i^n \right| - K_\alpha \tau^\alpha \| \delta_x^2 E^n \|^2 \\
&\leq \left| - \sum_{k=1}^{n-1} l_k \langle e^{-\rho k \tau} E^{n-k}, E^n \rangle \right| + \tau^\alpha h \left| \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i^n \right) R_i^n \right| - K_\alpha \tau^\alpha \| \delta_x^2 E^n \|^2 \\
&\leq \sum_{k=1}^{n-1} (-l_k) \left| \langle e^{-\rho k \tau} E^{n-k}, E^n \rangle \right| + \tau^\alpha h \left| \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i^n \right) R_i^n \right| - K_\alpha \tau^\alpha \| \delta_x^2 E^n \|^2 \\
&\leq -\frac{1}{2} \sum_{k=1}^{n-1} l_k \left( \langle E^n, E^n \rangle + \langle e^{-\rho k \tau} E^{n-k}, e^{-\rho k \tau} E^{n-k} \rangle \right) \\
&\quad - K_\alpha \tau^\alpha \| \delta_x^2 E^n \|^2 + \tau^\alpha h \left| \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i^n \right) R_i^n \right|
\end{aligned} \tag{3.16}$$

with the Cauchy-Schwarz inequality being used.

Let

$$B = \tau^\alpha h \left| \sum_{i=1}^{M-1} \left( \delta_x^2 \overline{E}_i \right) R_i^n \right|.$$

It is clear that

$$\begin{aligned} B &\leq \tau^\alpha h \sum_{i=1}^{M-1} \left| \left( \delta_x^2 \overline{E}_i \right) R_i^n \right| \\ &\leq \tau^\alpha h K_\alpha \sum_{i=1}^{M-1} \left| \delta_x^2 \overline{E}_i \right|^2 + \tau^\alpha h \frac{1}{4K_\alpha} \sum_{i=1}^{M-1} |R_i^n|^2 \\ &= K_\alpha \tau^\alpha \|\delta_x^2 E^n\|^2 + \frac{\tau^\alpha}{4K_\alpha} \|R^n\|^2. \end{aligned} \quad (3.17)$$

Since  $|R_i^n| \leq \tilde{C}(\tau + h^4)$ , it can be obtained that

$$\|R^n\|^2 \leq l \tilde{C}^2 (\tau + h^4)^2. \quad (3.18)$$

Substituting (3.17) into (3.16), and noticing (3.18), we have

$$l_0 \langle E^n, E^n \rangle \leq -\frac{1}{2} \sum_{k=1}^{n-1} l_k \langle E^n, E^n \rangle - \frac{1}{2} \sum_{k=1}^{n-1} l_k \langle E^{n-k}, E^{n-k} \rangle + \frac{C_1 \tau^\alpha}{2} (\tau + h^4)^2, \quad (3.19)$$

where  $C_1 = \frac{\tilde{C}^2 l}{2K_\alpha}$ . From Lemma 3.1, there exists

$$0 < -\frac{1}{2} \sum_{k=1}^{n-1} l_k < \frac{1}{2},$$

then it is derived immediately from (3.19) that

$$\langle E^n, E^n \rangle \leq -\sum_{k=1}^{n-1} l_k \langle E^{n-k}, E^{n-k} \rangle + C_1 \tau^\alpha (\tau + h^4)^2. \quad (3.20)$$

In the following, we prove

$$\langle E^n, E^n \rangle \leq \left( \sum_{k=0}^{n-1} l_k \right)^{-1} C_1 \tau^\alpha (\tau + h^4)^2 \quad (3.21)$$

by mathematical induction.

For  $n = 1$ , (3.21) holds by (3.20). Suppose

$$\langle E^s, E^s \rangle \leq \left( \sum_{k=0}^{s-1} l_k \right)^{-1} C_1 \tau^\alpha (\tau + h^4)^2$$

when  $s = 1, 2, \dots, n-1$ . Then for  $s = n$ , by (3.20) and the assumption we conclude that

$$\begin{aligned}
\langle E^n, E^n \rangle &\leq - \sum_{k=1}^{n-1} l_k \left( \sum_{m=0}^{n-k-1} l_m \right)^{-1} C_1 \tau^\alpha (\tau + h^4)^2 + C_1 \tau^\alpha (\tau + h^4)^2 \\
&\leq - \left( \sum_{k=0}^{n-1} l_k \right)^{-1} \sum_{k=1}^{n-1} l_k C_1 \tau^\alpha (\tau + h^4)^2 + C_1 \tau^\alpha (\tau + h^4)^2 \\
&= \left( 1 - \frac{\sum_{k=1}^{n-1} l_k}{\sum_{k=0}^{n-1} l_k} \right) C_1 \tau^\alpha (\tau + h^4)^2 \\
&= \left( \sum_{k=0}^{n-1} l_k \right)^{-1} C_1 \tau^\alpha (\tau + h^4)^2.
\end{aligned} \tag{3.22}$$

Finally, according to (3.22), (3.2), Lemma 2.1 and Lemma 2.2, we derive

$$\begin{aligned}
\|E^n\|_\infty^2 &\leq \frac{l}{4} \|\delta_x E^n\|^2 \\
&\leq \frac{3l}{8} \langle E^n, E^n \rangle \\
&\leq \frac{3l}{8} \left( \sum_{k=0}^{n-1} l_k \right)^{-1} C_1 \tau^\alpha (\tau + h^4)^2 \\
&\leq \frac{3l}{8} n^\alpha \Gamma(1-\alpha) C_1 \tau^\alpha (\tau + h^4)^2 \\
&\leq [C (\tau + h^4)]^2,
\end{aligned} \tag{3.23}$$

where  $C = \sqrt{\frac{3l}{8} T^\alpha \Gamma(1-\alpha) C_1}$ . □

#### 4. Numerical examples

In this section, we consider some numerical examples to demonstrate the effectiveness of the schemes, and verify the theoretical results including convergence orders and numerical stability. The discrete  $L^\infty$  norm is used to measure the numerical errors, which makes the results stronger than the discrete  $L^2$  norm does.

In the following examples, denote  $i = \sqrt{-1}$ , and we choose  $U(x) = 1$  and  $x$ , respectively.

**Example 4.1.** For the backward fractional Feynman-Kac equation (2.1) on the finite domain  $0 < x < 1$ ,  $0 < t \leq 1$ , take  $K_\alpha = 0.5$ ,  $U(x) = 1$ ,  $\rho = 1 + i$ , the forcing function

$$f(x, t) = \frac{\Gamma(4 + \alpha)}{\Gamma(4)} e^{-\rho t} t^3 \sin(\pi x) + K_\alpha \pi^2 (t^{3+\alpha} + 1) e^{-\rho t} \sin(\pi x),$$

with the initial condition  $P(x, 0) = \sin(\pi x)$ , and the boundary conditions  $P(0, t) = P(1, t) = 0$ . The exact solution is given by

$$P(x, t) = e^{-\rho t} (t^{3+\alpha} + 1) \sin(\pi x).$$

Table 1: The maximum errors and convergence orders for Example 4.1 in temporal direction with  $q = 1$  and  $h = 1/1000$ .

$\tau$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/10	0.0023	-	0.0080	-	0.0182	-
1/20	0.0011	1.0641	0.0041	0.9644	0.0093	0.9686
1/40	5.7830e-004	0.9276	0.0020	1.0356	0.0047	0.9846
1/80	2.9000e-004	0.9958	0.0010	1.0000	0.0024	0.9696

Table 2: The maximum errors and convergence orders for Example 4.1 in spatial direction with  $q = 1$  and  $\tau = h^4$ .

$h$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/2	0.0184	-	0.0217	-	0.0279	-
1/4	0.0011	4.0641	0.0013	4.0611	0.0017	4.0367
1/8	6.5910e-005	4.0609	7.8904e-005	4.0423	1.0316e-004	4.0426
1/10	2.6942e-005	4.0091	3.2266e-005	4.0074	4.2201e-005	4.0057

Table 1–Table 4 illustrate part of the numerical results to verify the convergence orders of schemes (2.10)–(2.12) for Example 4.1. Other cases also coincide with the results derived in Section 2, though we omit them here.

Table 3: The maximum errors and convergence orders for Example 4.1 in spatial direction with  $q = 2$  and  $\tau = h^2$ .

$h$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/10	2.7752e-005	-	3.5587e-005	-	5.1361e-005	-
1/20	1.7312e-006	4.0027	2.2245e-006	3.9998	3.2177e-006	3.9966
1/40	1.0815e-007	4.0007	1.3903e-007	4.0000	2.0122e-007	3.9992
1/80	6.7586e-009	4.0002	8.6896e-009	4.0000	1.2578e-008	3.9998

Table 4: The maximum errors and convergence orders for Example 4.1 in temporal direction with  $q = 3$  and  $h = 1/1000$ .

$\tau$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/10	2.4487e-005	-	9.5460e-005	-	2.4802e-004	-
1/20	3.0669e-006	2.9972	1.1984e-005	2.9938	3.1167e-005	2.9924
1/40	3.8372e-007	2.9987	1.5009e-006	2.9972	3.9044e-006	2.9968
1/80	4.7992e-008	2.9992	1.8778e-007	2.9987	4.8854e-007	2.9986
1/160	6.0077e-009	2.9979	2.3470e-008	3.0002	6.1094e-008	2.9994

**Example 4.2.** For the backward fractional Feynman-Kac equation (2.1) on the finite domain  $0 < x < 1$ ,  $0 < t \leq 1$ , take  $K_\alpha = 0.5$ ,  $U(x) = x$ ,  $\rho = 1 + i$ , the forcing function

$$f(x, t) = \frac{\Gamma(4 + \alpha)}{\Gamma(4)} e^{-\rho x t} t^3 \sin(\pi x) - K_\alpha e^{-\rho x t} (t^{3+\alpha} + 1) (\rho^2 t^2 \sin(\pi x) - 2\pi \rho t \cos(\pi x) - \pi^2 \sin(\pi x)),$$

with the initial condition  $P(x, 0) = \sin(\pi x)$ , and the boundary conditions  $P(0, t) = P(1, t) = 0$ . The exact solution is given by

$$P(x, t) = e^{-\rho x t} (t^{3+\alpha} + 1) \sin(\pi x).$$

Table 5–Table 11 show the maximum errors and convergence orders for Example 4.2 in both temporal and spatial directions, respectively, which confirm



Table 5: The maximum errors and convergence orders for Example 4.2 in temporal direction with  $q = 1$  and  $h = 1/1000$ .

$\tau$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/10	0.0038	-	0.0132	-	0.0301	-
1/20	0.0019	1.0000	0.0067	0.9783	0.0154	0.9668
1/40	9.5861e-004	0.9870	0.0034	0.9786	0.0078	0.9814
1/80	4.8072e-004	0.9957	0.0017	1.0000	0.0039	1.0000

Table 6: The maximum errors and convergence orders for Example 4.2 in spatial direction with  $q = 1$  and  $\tau = h^4$ .

$h$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/2	0.0771	-	0.0731	-	0.0713	-
1/4	0.0043	4.1643	0.0040	4.1918	0.0040	4.1558
1/8	2.6053e-004	4.0448	2.5475e-004	3.9728	2.6205e-004	3.9321
1/10	1.0733e-004	3.9742	1.0446e-004	3.9951	1.0689e-004	4.0186

Table 7: The maximum errors and convergence orders for Example 4.2 in temporal direction with  $q = 2$  and  $h = 1/1000$ .

$\tau$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/10	4.8365e-004	-	0.0018	-	0.0043	-
1/20	1.2599e-004	1.9407	4.6403e-004	1.9557	0.0011	1.9668
1/40	3.2132e-005	1.9712	1.1850e-004	1.9693	2.8775e-004	1.9346
1/80	8.1125e-006	1.9858	2.9935e-005	1.9850	7.2748e-005	1.9838
1/160	2.0381e-006	1.9929	7.5227e-006	1.9925	1.8288e-005	1.9920

the global truncation error of schemes (2.10)–(2.12) is  $\mathcal{O}(\tau^q + h^4)$  for  $q = 1, 2, 3, 4$ .

Table 8: The maximum errors and convergence orders for Example 4.2 in spatial direction with  $q = 2$  and  $\tau = h^2$ .

$h$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/10	1.0794e-004	-	1.0761e-004	-	1.1799e-004	-
1/20	6.7206e-006	4.0055	6.7031e-006	4.0048	7.3687e-006	4.0011
1/40	4.2056e-007	3.9982	4.1859e-007	4.0012	4.6134e-007	3.9975
1/80	2.6277e-008	4.0004	2.6157e-008	4.0003	2.8833e-008	4.0000

Table 9: The maximum errors and convergence orders for Example 4.2 in temporal direction with  $q = 3$  and  $h = 1/1000$ .

$\tau$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/10	4.0581e-005	-	1.5822e-004	-	4.1108e-004	-
1/20	5.0826e-006	2.9972	1.9862e-005	2.9938	5.1656e-005	2.9924
1/40	6.3591e-007	2.9987	2.4876e-006	2.9972	6.4710e-006	2.9969
1/80	7.9535e-008	2.9992	3.1123e-007	2.9987	8.0969e-007	2.9985
1/160	9.9561e-009	2.9979	3.8898e-008	3.0002	1.0126e-007	2.9993

Table 10: The maximum errors and convergence orders for Example 4.2 in spatial direction with  $q = 3$  and  $\tau = h^{4/3}$ .

$h$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/8	2.6076e-004	-	2.5785e-004	-	2.9742e-004	-
1/16	1.6025e-005	4.0243	1.5697e-005	4.0380	1.7149e-005	4.0262
1/32	1.0054e-006	3.9945	9.8854e-007	3.9890	1.0750e-006	3.9957
1/64	6.3848e-008	3.9770	6.2690e-008	3.9790	6.7897e-008	3.9848
1/128	3.9893e-009	4.0004	3.9163e-009	4.0007	4.2434e-009	4.0001

Table 11: The maximum errors and convergence orders for Example 4.2 in both temporal and spatial directions with  $q = 4$  and  $\tau = h$ .

$h$	$\alpha = 0.2$	Rate	$\alpha = 0.5$	Rate	$\alpha = 0.8$	Rate
1/10	1.0568e-004	-	9.7935e-005	-	8.9182e-005	-
1/20	6.5976e-006	4.0016	6.0897e-006	4.0074	5.5313e-006	4.0111
1/40	4.1219e-007	4.0006	3.8077e-007	3.9994	3.4481e-007	4.0037
1/80	2.5760e-008	4.0001	2.3789e-008	4.0006	2.1531e-008	4.0013
1/160	1.6098e-009	4.0002	1.4867e-009	4.0001	1.3451e-009	4.0006

## 5. Conclusion

In this paper, we construct compact finite difference schemes for solving the backward fractional Feynman-Kac equation by its equivalent form, where the  $q$ -th ( $q = 1, 2, 3, 4$ ) order approximation operators for fractional substantial derivative are used in temporal direction, and fourth order compact difference operator for the spatial derivative, respectively. By introducing a new inner product, we prove rigorously that the scheme is unconditionally stable and convergent in maximum norm when  $q = 1$ . For all the schemes proposed, from first to fourth order in temporal direction, abundant examples are performed to verify the theoretical analysis and their effectiveness.

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